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Dual solutions in mixed convection flow near a stagnation point on a vertical surface in a porous medium

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Abstract

The steady stagnation point flow through a porous medium bounded by a vertical surface is investigated in this study. The external velocity, which normally impinges the vertical surface and the surface temperature are assumed to vary linearly with the distance from the stagnation point. The governing system of partial differential equations is first transformed into a system of ordinary differential equations, and then they are solved numerically by a finite-difference scheme, namely the Keller-box method. The features of the flow and heat transfer characteristics for different values of the governing parameters are analyzed and discussed. Both cases of assisting and opposing flows are considered. It is found that dual solutions exist for assisting flow, besides that usually reported in the literature for opposing flow. Therefore, the reported results are completely new.

Keywords: Boundary layer; Dual solutions; Mixed convection; Stagnation point; Porous medium

1. Introduction

The study of fluid flow in a fibrous porous medium, such as filter, is important in many practical fields to understand the character of the porous medium in order to remove small particles contained in the fluid more effectively. For this study, Darcy's law may be used as a basic equation. This law expresses that the velocity is proportional to the pressure gradient and it does not contain the convective acceleration of the fluid. However, Darcy's law is considered valid for flows at low speed, whereas the speed in the filter is not always small and the convective force will become important, see Yamamoto and Iwamura [1]. The mathematical formulations based on Darcy's law will neglect the effects of a solid boundary or the inertia forces on fluid flow and heat transfer through porous media. In general, the inertia and boundary effects become significant when the fluid velocity is high and the heat transfer is considered in the near wall region, see Chen and Lin [2]. Brinkman [3] proposed a model which accounts for the transition from Darcy flow to highly viscous flow (without porous matrix), in the limit of extremely high permeability. However, Brinkman model does not account adequately for the transition from porous medium flow to pure fluid flow as the permeability of the porous medium increases. A model that bridges the entire gap between the Darcy and Navier–Stokes equations is the Darcy–Forchheimer model which was developed by Vafai and Tien [4] and which is very well described in the book by Nakayama [5]. Darcy–Forchheimer model with the velocity square inertial term neglected has been used by Raptis [6], Raptis and Perdikis [7], Raptis and Takhar [8], etc. to study the flow and heat transfer characteristics over vertical and horizontal flat plates, when the effects of suction and injection, as well as of magnetic field were included. This model has also been used by Du and Bilgen [9], and recently by Sathiyamoorthy et al. [10] to study the free convection in a square cavity.

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Nomenclature

a, b	constants
C_f	skin friction coefficient
f	dimensionless stream function
g	acceleration due to gravity
k	thermal conductivity
K	permeability parameter
K_1	permeability of the porous medium
Nu_x	local Nusselt number
Pe_x	local Péclet number
Pr	Prandtl number
q_w	heat transfer from the plate
Ra_x	local Rayleigh number
Re_x	local Reynolds number
T	fluid temperature
$T_w(x)$	plate temperature
T_∞	ambient temperature
u, v	velocity components along the x- and y-directions, respectively
$u_e(x)$	external velocity
x, y	Cartesian coordinates along the surface and normal to it, respectively
Greek symbols	
α	thermal diffusivity
β	thermal expansion coefficient
η	similarity variable
θ	dimensionless temperature
λ	buoyancy or mixed convection parameter
ν	kinematic viscosity
μ	dynamic viscosity
ρ	fluid density
τ_w	skin friction
ψ	stream function
Subscripts	
e	condition at the edge of the boundary layer
w	condition at the wall
∞	condition away from the wall
Superscript	
$'$	differentiation with respect to η

In the present paper, we study the mixed convection flow near the stagnation point on a vertical surface which is embedded in a fluid-saturated porous medium using the Darcy–Forchheimer model with the velocity square term neglected. It is shown that dual solutions exist in the opposing flow regime and they continue into that of the assisting flow regime, i.e. when the buoyancy force acts in the same direction as the inertia force. To this end, we wish to mention that only a few attempts have been made to find the second solutions for the assisting flow case. Ridha [11] probably is the first to find the existence of dual solutions in both cases, assisting and opposing flows when he studied the problems of mixed-convection flow over a horizontal surface, mixed-convection flow over a vertical surface and axisymmetric mixed-convection on a vertical cylinder. Very recently, Merrill et al. [12] improved the result obtained by Nazar et al. [13] for purely Darcy flow, showing that the lower branch solutions exist in the opposing flow regime and they continue into the assisting flow regime, while the latter reported the existence of dual solutions only in the case of opposing flow. Merrill et al. [12] found that dual solutions exist for all values of the buoyancy parameter $\lambda > \lambda_c$, where λ_c (<0) is the value of λ for which the upper branch solution meets the lower branch solution.

2. Analysis

Consider the steady laminar boundary layer flow towards the stagnation point on a vertical flat plate, which is embedded in a porous medium as shown in Fig. 1. Either heating or cooling of the plate is assumed to begin simultaneously with the motion of the external stream. It is assumed that the temperature of the plate, $T_w(x)$, and the external velocity, $u_e(x)$, vary linearly with the distance from the stagnation point. It is also assumed that the temperature of the ambient fluid is T_∞, where $T_w(x) > T_\infty$ for a heated plate (assisting flow) and $T_w(x) < T_\infty$ for a cooled plate (opposing flow). The physical properties of the fluid are assumed to be constant except the density in the buoyancy force term, which is satisfied by the Boussinesq approximation. We shall use the Darcy–Forchheimer model, where the velocity square term is neglected. Under these assumptions, the boundary layer equations are (see Vafai and Tien [4] or Nakayama [5])

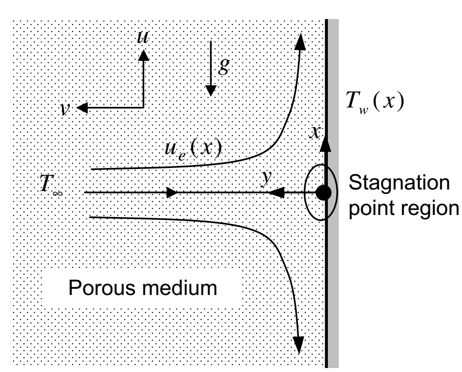

Fig. 1. Physical model and coordinate system.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{du_e}{dx}+v\frac{\partial^2 u}{\partial y^2}+\frac{v}{K_1}(u_e-u)+g\beta(T-T_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$u=0,\quad v=0,\quad T=T_w,\quad \text{at } y=0,$$
$$u\to u_e(x),\quad T\to T_\infty\quad \text{as } y\to\infty, \tag{4}$$

where x and y are the Cartesian coordinates with the origin at the stagnation point along and normal to the plate, respectively, u and v are the velocity components along the x- and y-axes, respectively, T is the fluid temperature and the other physical quantities are defined in the Nomenclature.

We assume that the external velocity $u_e(x)$ and the plate temperature $T_w(x)$ are respectively given by

$$u_e(x)=ax,\quad T_w(x)=T_\infty+bx, \tag{5}$$

where a (>0) and b are constants with $b>0$ for an assisting flow and $b<0$ for an opposing flow (see Sparrow et al. [14]).

We introduce now the following similarity variables (see Cheng [15] or Lai and Kulacki [16])

$$\eta=\left(\frac{u_e x}{\alpha}\right)^{1/2}\frac{y}{x},\psi=(\alpha u_e x)^{1/2}f(\eta),\quad \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \tag{6}$$

where ψ is the stream function defined as $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$, so as to identically satisfy Eq. (1). Substituting (6) into Eqs. (2) and (3), we get the following ordinary differential equations:

$$Prf'''+ff''+1-f'^2+K(1-f')+\lambda K\theta=0, \tag{7}$$
$$\theta''+f\theta'+f'\theta=0, \tag{8}$$

where primes denote differentiation with respect to η, $Pr=v/\alpha$ is the Prandtl number, $K=v/(aK_1)$ is the permeability parameter and $\lambda=Ra_x/Pe_x$ is the buoyancy parameter, with $Ra_x=K_1g\beta(T_w-T_\infty)x/(v\alpha)$ and $Pe_x=u_ex/\alpha$ being the local Rayleigh number and local Péclet number, respectively. We notice that λ is a constant with $\lambda>0$ corresponds to assisting flow and $\lambda<0$ corresponds to opposing flow, while $\lambda=0$ is for forced convection limit. The boundary conditions (4) become

$$f(0)=f'(0)=0,\quad \theta(0)=1,\quad f'(\infty)\to 1,\quad \theta(\infty)\to 0. \tag{9}$$

It is interesting to note that when $K\to\infty$ (pure Darcy flow), Eq. (7) reduces to

$$f'=1+\lambda\theta, \tag{10}$$

along with the boundary conditions for Eqs. (8) and (10)

$$f(0)=0,\theta(0)=1,\theta(\infty)\to 0. \tag{11}$$

Eqs. (8), (10) and (11) are similarity equations for the case of pure Darcy flow previously considered by Lai and Kulacki [16] for impermeable plate, in their paper. Therefore, the results reported in [16] can be used for comparison, which can support the validity of the present results. Further, Eqs. (8) and (10) can be combined to obtain

$$f'''+ff''+f'-f'^2=0, \tag{12}$$

subject to the boundary conditions

$$f(0)=0,\quad f'(0)=1+\lambda,\quad f'(\infty)\to 1. \tag{13}$$

Eq. (12) subjected to (13) is the governing equation for the final steady flow previously solved by Nazar et al. [13], where the existence of dual solutions was reported in the range $-1.4175<\lambda<-1$. Recently, Merrill et al. [12] improved this result, showing that dual solutions exist for all values of $\lambda>-1.4175$. In this study, we show that for $K=1$, 10, 100 and 1000, dual solutions exist in the opposing flow regime and they continue into that of assisting, which agree with Merrill et al. [12], for the case $K\to\infty$.

The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined by

$$C_f=\frac{\tau_w}{\rho u_e^2/2},\quad Nu_x=\frac{xq_w}{k(T_w-T_\infty)}, \tag{14}$$

where the skin friction τ_w and the heat transfer from the plate q_w are given by

$$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0},\quad q_w=-k\left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{15}$$

with μ and k being the dynamic viscosity and thermal conductivity, respectively. Using the similarity variables (6), we get

$$\frac{1}{2}C_fRe_x^{1/2}/Pr^{1/2}=f''(0),\quad Nu_x/Pe_x^{1/2}=-\theta'(0), \tag{16}$$

where $Re_x=u_ex/v$ is the local Reynolds number.

3. Results and discussion

The system of equations (7)–(9) has been solved numerically for some values of the permeability parameter K and buoyancy parameter λ, while the Prandtl number Pr is fixed to be unity ($Pr=1$), except for comparisons with previously reported cases. The nonlinear ordinary differential equations have been solved numerically using the Keller-box method by integrating forwards in η until a predetermined large value of η is reached, η_∞ say, where we assume the infinity boundary condition may be enforced. The step size of η, $\Delta\eta$, and the edge of the boundary layer, η_∞, are adjusted for different range of parameters. The Keller-box method is very well described in the book by Cebeci and Bradshaw [17].

The values of the local Nusselt number in terms of $-\theta'(0)\sqrt{Pr}$ and $-\theta'(0)$ are obtained and compared with previously reported cases. The comparisons are shown in Tables 1 and 2, respectively. It is seen that the values of $-\theta'(0)\sqrt{Pr}$ obtained from this study are in very good agreement with the results reported by Yih [18], and the values of $-\theta'(0)$ are in excellent agreement with the results given by Lai and Kulacki [16] and Yih [19]. Therefore, it can be concluded that the developed code can be used with great confidence to study the problem discussed in this paper. In addition, Table 2 also presents the second value of $-\theta'(0)$ for a particular value of λ ($\neq 0$) that neither reported by Lai and Kulacki [16] nor Yih [19].

The variations of the skin friction coefficient $f''(0)$ and the local Nusselt number $-\theta'(0)$ with buoyancy parameter λ for $K = 10$ and 100 are shown in Figs. 2 and 3 respectively, all for $Pr = 1$. These figures show that dual solutions exist in the opposing flow regime ($\lambda < 0$) and they continue into the assisting flow regime ($\lambda > 0$). For $\lambda > 0$, there is a favorable pressure gradient due to the buoyancy forces, which results in the flow being accelerated and consequently there is a larger skin friction coefficient than in the non-buoyant case ($\lambda = 0$). For negative values of λ, there is a critical value λ_c, with two branches of solution for $\lambda > \lambda_c$, a saddle-node bifurcation at $\lambda = \lambda_c$ and no solutions for $\lambda < \lambda_c$. The boundary layer separates from the surface at $\lambda = \lambda_c$, thus we are unable to get the solution for $\lambda < \lambda_c$ by using the boundary layer approximations. To obtain the solutions beyond this value, the full Navier–Stokes equations have to be used. Based on our computations, $\lambda_c = -1.7002$ for $K = 10$, and $\lambda_c = -1.4599$ for $K = 100$. The boundary layer separation occurs at $\lambda = \lambda_c$ where $f''(0) < 0$, a different result from the classical boundary layer theory where separation occurs when $f''(0) = 0$.

We identify the upper and lower branch solutions in the following discussion by how they appear in Fig. 2, i.e. the upper branch solution has a higher value of $f''(0)$ for a given λ than the lower branch solution. For assisting flow, dual solutions are found to exist for all positive values of λ considered, to much higher values than shown in Fig. 2. This figure also shows that the critical value $|\lambda_c|$ increases as the permeability parameter K is decreased, suggesting that higher porosity of the porous medium (small values of K) increases the range of existence of solutions to Eqs. (7)–(9), i.e. the boundary layer separation can be delayed by increasing the porosity of the medium. The

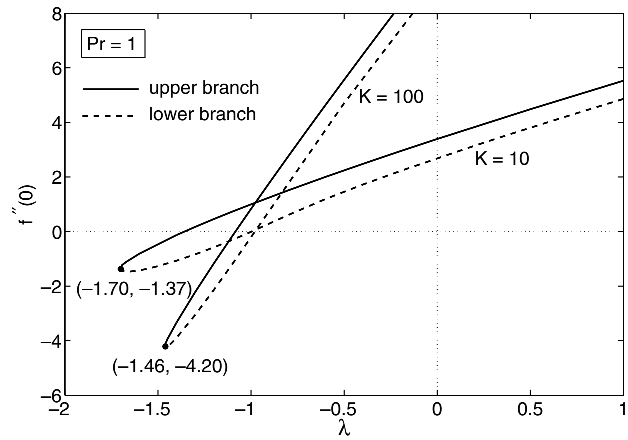

Fig. 2. Skin friction coefficient $f''(0)$ as a function of λ for $K = 10$ and 100 when $Pr = 1$.

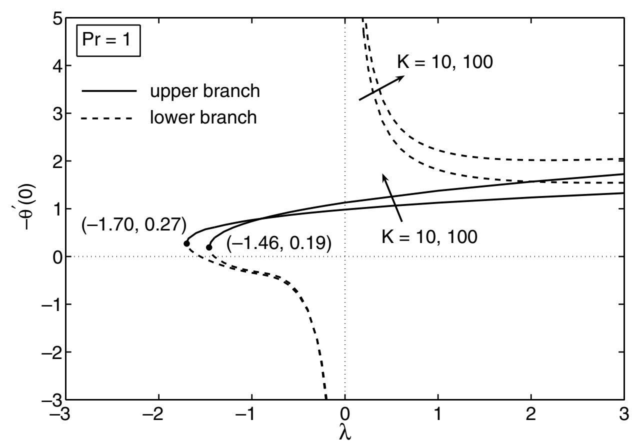

Fig. 3. Local Nusselt number $-\theta'(0)$ as a function of λ for $K = 10$ and 100 when $Pr = 1$.

Table 1
Values of $-\theta'(0)\sqrt{Pr}$ for various values of Pr when $\lambda = 0$ and $K = 0$

Pr	Yih [18]	Present results
0.0001	0.012433	0.0124
0.001	0.038658	0.0387
0.01	0.116372	0.1164
0.1	0.324927	0.3249
1	0.811301	0.8113
10	1.861577	1.8616
100	4.115021	4.1150
1000	8.963783	8.9638
10000	19.408995	19.4090

Table 2
Values of $-\theta'(0)$ for various values of λ when $Pr = 1$ and $K = 10^{10}$

λ	Lai and Kulacki [16]	Yih [19]	Present results	
			Upper branch	Lower branch
−1.0	0.7314	0.7314	0.7314	−0.3027
−0.8	0.8640	0.8640	0.8640	−0.3666
−0.6	0.9772	0.9771	0.9771	−0.5644
−0.4	1.0776	1.0776	1.0776	−1.1789
−0.2	1.1690	1.1690	1.1690	−3.3454
0.0	1.2533	1.2533	1.2533	–
0.5	1.4419	1.4419	1.4419	3.3713
1.0	1.6078	1.6078	1.6078	2.6356
3.0	2.1446	2.1445	2.1445	2.5817
10.0	3.4081	3.4081	3.4081	3.6458
20.0	4.6499	4.6499	4.6499	4.8503

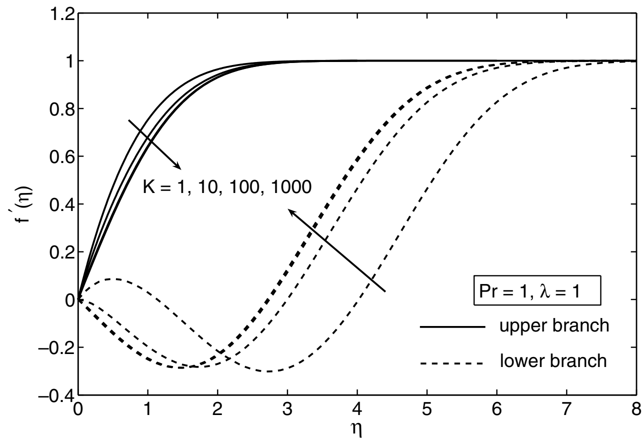

Fig. 4. Velocity profiles $f'(\eta)$ for different values of K when $Pr = 1$ and $\lambda = -1$ (opposing flow).

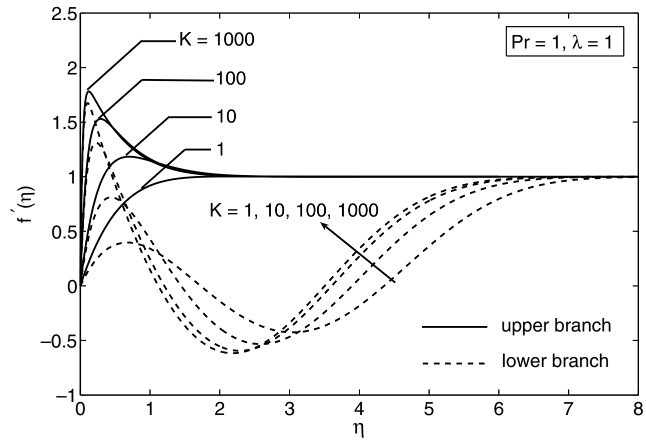

Fig. 6. Velocity profiles $f'(\eta)$ for different values of K when $Pr = 1$ and $\lambda = 1$ (assisting flow).

results shown in Fig. 3 for the heat transfer rate at the surface, $-\theta'(0)$, suggest that for the lower branch solution, $-\theta'(0)$ becomes unbounded as $\lambda \to 0^-$ and as $\lambda \to 0^+$.

Figs. 4 and 5 respectively illustrate the samples of velocity and temperature profiles for opposing flow, $\lambda = -1$, while the corresponding assisting flow, $\lambda = 1$, are shown in Figs. 6 and 7, all for $Pr = 1$. In these figures the solid lines are for the upper branch solutions and the dash lines for the lower branch solutions. As seen in Figs. 4–7, there are dual solutions both when $\lambda = -1$ and $\lambda = 1$. The dual solutions are obtained by setting different values of η_∞, as can be seen from Figs. 4–7 that the lower branch solutions have larger boundary layer thickness compared to the upper branch solutions. To which of the dual solutions, the flow will approach as it developes from the leading edge depends essentially on the stability of the solution. A full stability analysis is beyond the scope of the present work, and is an importance challenge for future research. However, we suggest that it will be the upper branch solutions that are the most physical relevance, since the forced convection limit ($\lambda = 0$) is on the upper branch and it is the only solution for this case. Although the lower branch solutions seem to deprive physical significance, they are nevertheless of interest in so far as the differential equations are concerned. Similar equations may reappear in other situations where the corresponding solutions have more realistic meaning.

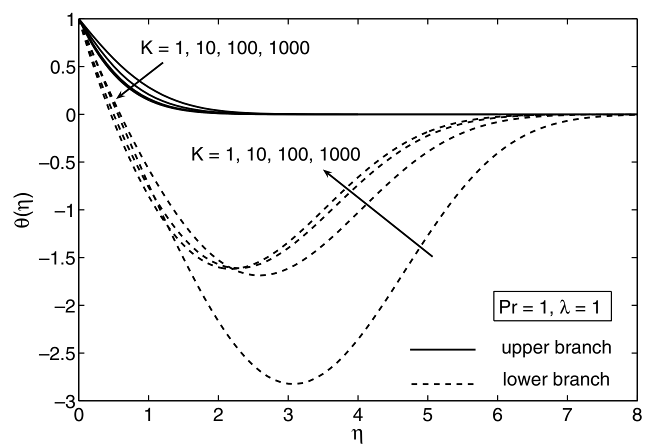

Fig. 7. Temperature profiles $\theta(\eta)$ for different values of K when $Pr = 1$ and $\lambda = 1$ (assisting flow).

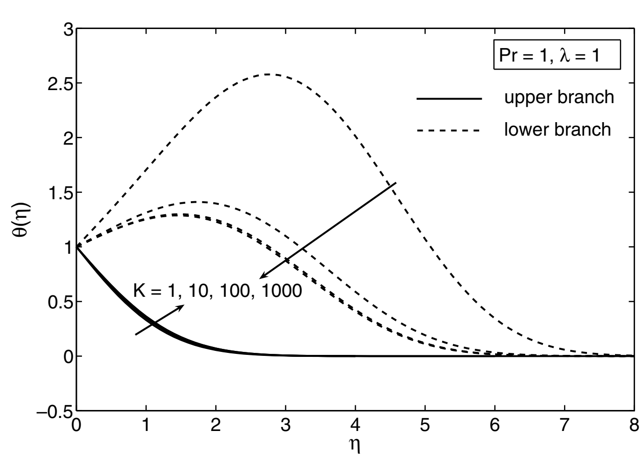

Fig. 5. Temperature profiles $\theta(\eta)$ for different values of K when $Pr = 1$ and $\lambda = -1$ (opposing flow).

Finally, Figs. 4–7 show that the boundary conditions (9) are satisfied, which support the validity of the present results, besides support the dual nature of the solution to the boundary-value problem (7)–(9).

4. Conclusions

We have theoretically studied the similarity solutions for steady stagnation point flow through a porous medium bounded by a vertical plate immersed in an incompressible viscous fluid using the generalized equation of Darcy's law. The governing nonlinear ordinary differential equations were solved numerically using the Keller-box method. We discussed the effects of the permeability parameter K and the buoyancy parameter λ on the fluid flow and heat transfer characteristics, while the Prandtl number is fixed to be unity.

A novel feature to emerge from the present study is the existence of a reversed flow region, in addition to a dual-solution in the assisting flow regime ($\lambda > 0$), besides that in the opposing flow regime ($\lambda < 0$). In the assisting flow case, solutions could be obtained for all positive values of λ, whereas in the opposing case the solution terminates with a saddle-node bifurcation at $\lambda = \lambda_c$ ($\lambda_c < 0$). The value of $|\lambda_c|$ decreases with an increase in K, thus increasing K is to accelerate the boundary layer separation, which in turn decreases the range of similarity solutions.

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